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Modified Upside Down Bathtub-Shaped Hazard Function Distribution: Properties and Applications

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Abstract:
A modified upside-down bathtub-shaped hazard function distribution that we have recommended in this study is created by modifyinga three-parameter life-time distribution. The skewness, and kurtosis measures, quantile function, survival function, hazard rate function, the probability density function, and cumulative distribution function are only a few of the mathematical and statistical properties of the distribution that we have been covered.Utilizing MLE, LSE, and CVME approaches, the model parameters of the suggested model are estimated. The suggested model's goodness of fit is also assessed by fitting it against a number of other life-time models using two real data sets.

Keywords: Modified distribution, Maximum likelihood estimation, Quantile function, Reliability function,

1. Introduction

Lifetime distributions are frequently used in reliability and survival studies to measure the average lifespan of system and device components. In disciplines including biological science, information technology, engineering, insurance, etc., lifetime distributions are often employed. In statistical literature, a wide variety of continuous probability distributions, including Cauchy, exponential, gamma, and Weibull, have frequently been employed to assess lifetime data.The modified distributions are often very helpful to investigate additional characteristics of the events that cannot be explored by classical distributions. Many modified distributions have been created by one or more shape parameters and found that they are flexible to analyze life-time datasets (Sheikh et al., 1987). Some of the well-known modified models are found in literature as (Rasekhi et al., 2017) have defined the modified exponential distribution whose hazard rate function (hrf) has increasing and S-shaped. Similarly two-parameter modified weighted exponential distribution has defined by (Chesneau et al., 2022) by mixing exponential and weighted exponential distributions. Lai et al., (2003) have defined a modified Weibull distribution having a bathtub-shaped failure-rate function derived as a limiting case of the Beta Integrated Model. Another modification of the Weibull distribution was introduced by (Sarhan & Zaindin, 2009) and named it modified Weibull distribution and new modified Weibull distribution was defined (Almalki& Yuan, 2013). A five-parameter model called the beta modified Weibull distribution is recommended by (Silva, et al., 2010) having a monotone,

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unimodal and bathtub-shaped hazard functions.A reduced new modified Weibull distribution is introduced by (Almalki, 2018). The Kumaraswamy modified Weibull distribution is recommended by (Cordeiro et al., 2014), Poisson modified Weibull distribution (Abd El□Monsef et al., 2022) and modified inverse NHE distribution (Chaudhary et al., 2022).

In the study of survival and reliability of a component or event or a system, we may encounter with three-step behavior of the failure rate will be observed at that situation a distribution with a bathtub and upside down bathtub-shaped (UDB) failure rate would be appropriate (Rajarshi & Rajarshi, 1988). With an increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped failure rate, Dimitrakopoulou et al., (2007) presented a three-parameter life-time distribution. The hazard and probability density functions (pdf) of UBD distribution is in the form of,

$$h(x) = \alpha\beta\lambda x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}; x>0, (\alpha,\beta,\lambda)>0$$

$$f(x) = \alpha\beta\lambda x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}\exp\left\{1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right\}; x>0, (\alpha,\beta,\lambda)>0 \quad (1)$$

It is a special case of Weibull distribution, when α = 1, it reduces to Weibull distribution. In this study we have modifiedequation (1) to introduce proposedmodel.

The goal of this study is to propose a more flexible model that can have a failure rate that is shaped like a bathtub, an upside-down bathtub, an increasing or decreasing, with the fewest possible parameters.The residualsections of the suggested model are structured as follows. We suggest a novel distribution and discuss various distributional properties in section 2.LSE), MLE, and CVME approaches are three popular estimation techniques that we have taken into consideration when estimating the parameters of the suggested distribution.In section 3,using the observed information matrix, we have created asymptotic confidence intervals for the maximum likelihood (ML) estimate. A real data set has been examined in section 4 to examine the potential uses and capabilities of the suggested distribution. The recommendedmodel's goodness of fit is assessed by fitting it to a real data set and comparing it to a few other existing distributions. Finally, we offer some concluding remarks in section 5.

2. New distribution

A three-parameter new modified upside down bathtub-shaped hazard function (MUBD) distribution is introduced by modifying the distribution defined by (Dimitrakopoulou et al., 2007).

Cumulative distribution function (CDF) of MUBD:

The CDF of MUBD model with parameters α, β and λ is

$$F(x) = 1-exp\left\{1-\left(1+x^{\beta}e^{-\lambda/x}\right)^{\alpha}\right\}; \quad \alpha>0,\ \beta>0, \lambda>0 \text{ and } x \in (0,\infty) \quad (2)$$

The probability density function (PDF):

The associatedPDF of MUBD distribution is

$$f(x) = \alpha\left(\beta+\lambda/x\right)x^{(\beta-1)}e^{-\lambda/x}\left(1+x^{\beta}e^{-\lambda/x}\right)^{\alpha-1} exp\left\{1-\left(1+x^{\beta}e^{-\lambda/x}\right)^{\alpha}\right\} \quad (3)$$

Survival Function:

$$S(x) = exp\left\{1-\left(1+x^{\beta}e^{-\lambda/x}\right)^{\alpha}\right\}$$

Hazard Function:

$$H(x) = \alpha(\beta+\lambda/x)x^{-(\beta+1)}e^{-\lambda/x}\left(1+x^{\beta}e^{-\lambda/x}\right)^{\alpha-1};\ \alpha > 0,\ \beta>0, \lambda>0 \quad (4)$$

The Quantile function:

$$\left[log\left\{\left[1-\left(log\left(1-p\right)\right)^{1/\alpha}\right]-1\right\}\right] - \beta\, log\, x + \lambda / x = 0; 0 < p < 1 \quad (5)$$

Solving equation (5) for x we will get the quantile function where p follows uniform distribution [0, 1].

Skewness and Kurtosis

Based on quartiles, the coefficient of skewness can be obtained using the expression

$$S_k = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1} \quad (6)$$

where Q_1, Q_2 *and* Q_3 are lower, median and lower quartiles respectively.

Moors (1988) presented the concept that the kurtosis coefficient depends on the octiles, and it may be written as

$$K_M = \frac{Q(0.375) - Q(0.125) - Q(0.625) + Q(0.875)}{Q(0.75) - Q(0.25)} \quad (7)$$

Figure 1 displays plots of the suggested distribution's probability density function and hazard rate function for various parameter values.

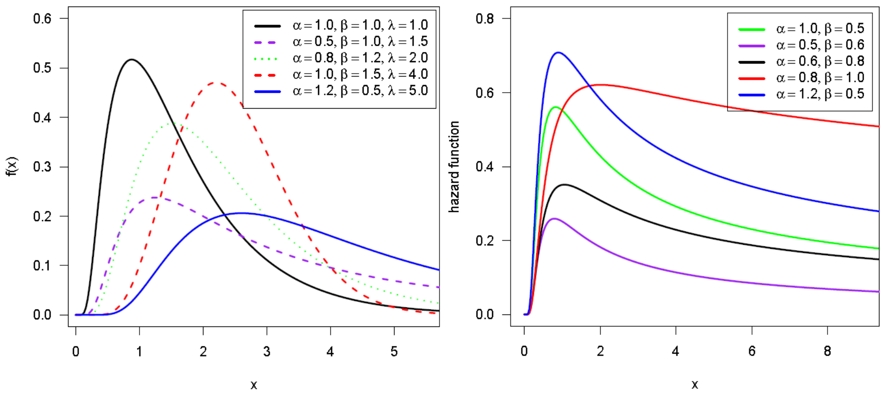


Figure 1. Hazard function (right part) and density function (left part) shapes of numerous α, β and λvalues.

3. Parameter estimation methods

a) Method of Maximum Likelihood Estimation (MLE)

Let a random sample with size 'n' drawnfrom the proposed model be $\underline{x} = (x_1,\ldots,x_n)$, then the log likelihood function can be expressed as,

$$\ell = n\ln\alpha + \sum_{i=1}^{n} ln(\beta+\lambda/x_i) - (\beta+1)\sum_{i=1}^{n} ln\, x_i - \sum_{i=1}^{n}\lambda/x_i + (\alpha-1)\sum_{i=1}^{n} ln\{W(x_i)\} + n - \sum_{i=1}^{n}\{W(x_i)\}^{\alpha} \quad (8)$$

Where $W(x_i) = 1 + x_i^{\beta} e^{-\lambda/x_i}$

By differentiating (8) with respect to β ,λ andα, we have

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} lnW(x_i) - \sum_{i=1}^{n} W(x_i)^{\alpha}\, lnW(x_i)$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=1}^{n} ln\, x_i - (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-\lambda/x_i}\, ln\, x_i}{W(x_i)} - \alpha\sum_{i=1}^{n} W(x_i)^{\alpha-1} x_i^{\beta} e^{-\lambda/x_i}\, ln\, x_i$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\frac{x_i}{\beta+\lambda/x_i} - \sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\beta-1}e^{-\lambda/x_i}}{W(x_i)} - \alpha\sum_{i=1}^{n} x_i^{\beta-1}e^{-\lambda/x_i}W(x_i)^{\alpha-1}$$

We can get the ML estimators of the proposed distribution by setting these non-linear equations to zero and figuring out solutions for the unknown parameters (α, β, λ). Since it is impossible to calculate these equations manually, one can solve them by using the suitable computer software. Let $\underline{\Delta} = (\alpha,\beta,\lambda)$ be the parameter vector and the related maximum likelihood estimation for $\underline{\Delta}$ be represented by $\hat{\underline{\Delta}} = (\hat{\alpha},\hat{\beta},\hat{\lambda})$, then resulting asymptotic normality is $\left(\hat{\underline{\Delta}} - \underline{\Delta}\right) \to N_3\left[0,\left(K(\underline{\Delta})\right)^{-1}\right]$.Here, Fisher's information matrix is denoted by $K(\underline{\Delta})$ which is given by,

$$K(\underline{\Delta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) \end{pmatrix}$$

The MLE's asymptotic variance $\left(K(\underline{\Delta})\right)^{-1}$ is meaningless because in actuality we don't know $\underline{\Delta}$. The estimated parameter values are therefore plugged in to approximate the asymptotic variance. The information matrix $K(\underline{\Delta})$ is estimated by the following observed fisher information matrix $O(\hat{\underline{\Delta}})$.

$$O(\hat{\underline{\Delta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\lambda} \\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\beta\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2} \end{pmatrix}_{\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)} = -H(\underline{\Delta})_{\left(\underline{\Delta}=\hat{\underline{\Delta}}\right)}$$

where H denotes the Hessian matrix.

The observed information matrix is produced by Newton-Raphson approach with the aim of maximization of likelihood. As a result, variance-covariance matrix can be provided through,

$$\left[-H(\Delta)_{|(\Delta=\hat{\Delta})}\right]^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\beta}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix} \tag{9}$$

So,by MLEs'asymptotic normality, the following can be used to create approximate 100(1-δ) % confidence intervals for calculating α, β and λ:

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{\mathrm{var}(\hat{\alpha})}\ ,\ \hat{\beta} \pm Z_{\delta/2}\sqrt{\mathrm{var}(\hat{\beta})}\ \text{and}\ \hat{\lambda} \pm Z_{\delta/2}\sqrt{\mathrm{var}(\hat{\lambda})} \tag{10}$$

Here, $Z_{\delta/2}$ represents the upper percentile of standard normal variate.

b) Least-square estimation(LSE) method

The estimation of α, β and λ of MUBDmodel employing least-square estimation method can be found by minimizing (11) with regard to α, β and λ.

$$D(x;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2 \tag{11}$$

AssumeCDFfor the ordered random variables $x_{(1)} < \ldots < x_{(n)}$ be $F(X_i)$ .Here, $\underline{X} = (X_1,\ldots,X_n)$ represents a random sample with size n drawn from a distribution function F (.).Thus, by minimizing (12) with regard to α, β and λ, the LS estimators of α, β and λmay be determined.

$$D(\underline{X};\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[1-\exp\left\{1-W(x_{(i)})^{\alpha}\right\} - \frac{i}{n+1}\right]^2; \alpha > 0, \beta > 0, \lambda > 0. \tag{12}$$

Where $W(x_{(i)}) = 1 + x_{(i)}^{\beta} e^{-\lambda/x(i)}$.

Differentiating (12) with regard to α, β and λ

$$\frac{\partial D}{\partial \alpha} = 2\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\} - \frac{i}{n+1}\right] exp\left\{1-W(x_{(i)})^{\alpha}\right\} W(x_{(i)})^{\alpha}\, lnW(x_{(i)})$$

$$\frac{\partial D}{\partial \beta} = 2\alpha\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\} - \frac{i}{n+1}\right] x_{(i)}^{\beta} e^{-\lambda/x_{(i)}}\, ln\left(x_{(i)}\right) exp\left\{1-W(x_i)^{\alpha}\right\} W(x_i)^{\alpha-1}$$

$$\frac{\partial D}{\partial \lambda} = -2\alpha\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\} - \frac{i}{n+1}\right] x_{(i)}^{\beta-1} e^{-\lambda/x_{(i)}}\, exp\left\{1-W(x_{(i)})^{\alpha}\right\} W(x_{(i)})^{\alpha-1}$$

Therefore, by simultaneously solving the three equations above, the LS estimators forα, β and λ can be obtained.

c) Cramer-Von-Mises estimation(CVME) Method

By minimizing the function (13), the CVMEs of α, β and λ can be found.

$$A(\underline{X};\alpha,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\}-\frac{2i-1}{2n}\right]^2 \quad (13)$$

Differentiating (13) with regard to α, β and λ

$$\frac{\partial A}{\partial\alpha}=2\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\}-\frac{2i-1}{2n}\right]exp\left\{1-W(x_{(i)})^{\alpha}\right\}W(x_{(i)})^{\alpha}\,lnW(x_{(i)})$$

$$\frac{\partial A}{\partial\beta}=2\alpha\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\}-\frac{2i-1}{2n}\right]x_{(i)}^{\beta}e^{-\lambda/x_{(i)}}\,ln\left(x_i\right)exp\left\{1-W(x_{(i)})^{\alpha}\right\}W(x_{(i)})^{\alpha-1}$$

$$\frac{\partial A}{\partial\lambda}=-2\alpha\sum_{i=1}^{n}\left[1-exp\left\{1-W(x_{(i)})^{\alpha}\right\}-\frac{2i-1}{2n}\right]x_{(i)}^{\beta-1}e^{-\lambda/x_{(i)}}\,exp\left\{1-W(x_{(i)})^{\alpha}\right\}W(x_{(i)})^{\alpha-1}$$

We will get the CVM estimators after solving the following non-linear equationssimultaneously.

$$\frac{\partial A}{\partial\alpha}=0,\frac{\partial A}{\partial\beta}=0\ and\ \frac{\partial A}{\partial\lambda}=0 \quad (14)$$

4. Application to Real Dataset

In this section, we have used two real datasets from previous research to show how the MUBD can be applied. .

Dataset-1

This real data set is used by (Bader & Priest, 1982).

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426,2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

The likelihood function (8) is maximized to determine the MLEs by employing the **`maxLik()`** function in R software (R Core Team, 2022) and (Wickham &Grolemund, 2016)). Log-Likelihood value acquired is l = -48.8512, which we have attained and the LSE's, MLE's and CVME's for α, β, and λ are displayed in Table 1. Further we have calculated the Akaike information criterion (AIC) and Kolmogorov-Smirnov (KS) test statistics for three estimation approaches.

Table 1

Log-likelihood, Estimated parameters, AIC and Kolmogorov-Smirnov (KS) statistics

Estimation Method	alpha	Beta	lambda	LL	AIC	KS(p-value)
MLE	0.8559	3.0133	7.0336	-48.8512	103.7024	0.0407(0.9996)
LSE	0.8772	3.0125	7.1252	-48.8656	103.7312	0.0387(0.9998)
CVME	0.8735	3.0876	7.2803	-48.9455	103.8911	0.0362(0.9999)

Table 1 shows that the estimated parameters as well as AIC and p-values corresponding to different methods of estimation has less variation.
For the ML estimations α, β and λ, Figure 2 displays the profile log-likelihood function graphs. We discover that the three ML estimationsβ, λ, andα can each be individually created.

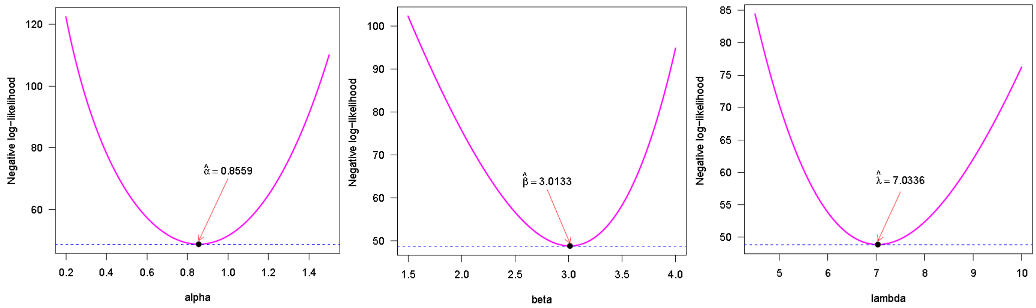


Figure. 2 ML estimations of α, β, and λ represented as profile log-likelihood function graphs

We frequently make use of the PDF and CDF plots to assess the goodness of fit of a suggestedmodel. We must plot Q-Q and P-P graphs in order to obtain the additional information.The P-P plot emphasizes the lack of fit, whereas Q-Q plot may deliver information on lack of fit towards distribution's tails. The excellent fit of the MUBD model to the data is shown in Figure 3.

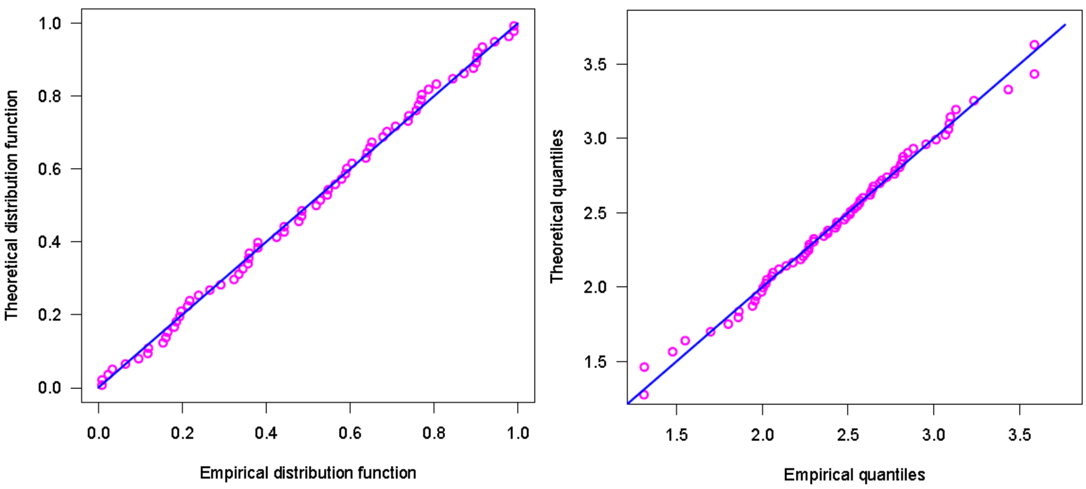


Figure 3. TheQ-Q graph (right part) and P-P graph (left part) for MUBDmodel.

We have chosen a few well-known distributions for comparison in order to show the MUBD distribution's goodness of fit. These are Modified Weibull (MW)(Lai et al., 2003),Exponentiated Exponential Poisson (EEP)(Ristić&Nadarajah, 2014),Generalized Exponential Extension (GEE) distribution (Lemonte, 2013),Weibull Extension Model (Tang et al., 2003) and Generalized Exponential (GE) distribution (Gupta &Kundu, 1999).

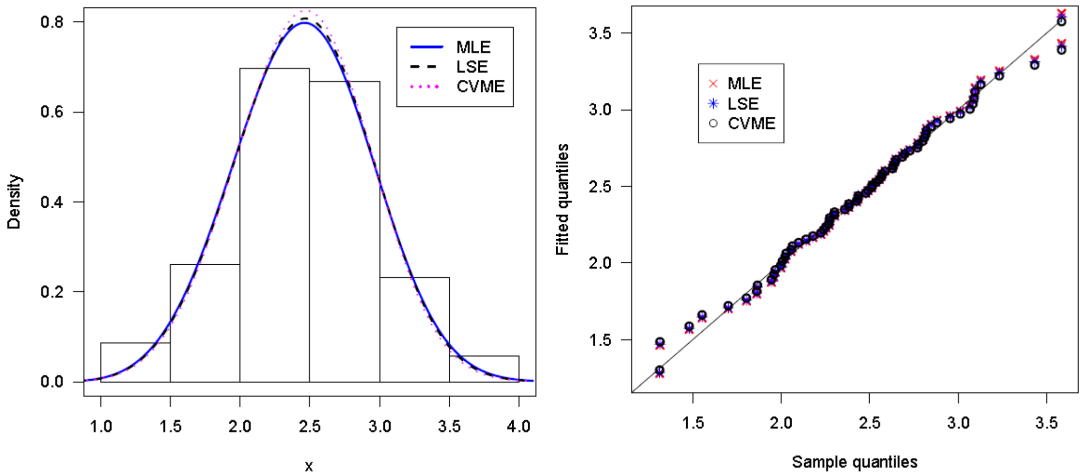


Figure 4.Q-Q plot (right part) of MLE, LSE, and CVM, as well as a histogram and the density function of fitted distributions (left part).

In order to evaluate the potential of the proposed model, the values of the Corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Akaike information criterion (AIC), and Bayesian information criterion (BIC) are computed. The findings are displayed in Table 2.

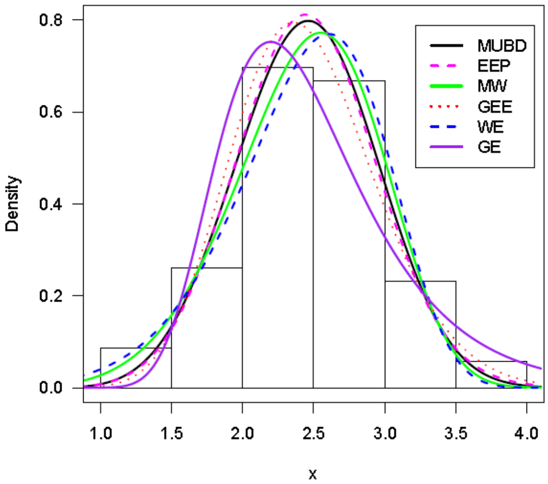
Table 2

AIC, BIC, CAIC, HQIC, and Log-likelihood (LL) of MUBD distribution

Distributions	LL	AIC	BIC	CAIC	HQIC
MUBD	-48.8512	103.7024	110.4047	104.0716	106.3614
EEP	-48.8574	103.7148	110.4172	104.0841	106.3741
WE	-49.6053	105.2106	111.9129	105.5798	107.8696
GEE	-50.4490	106.8981	113.6004	107.2673	109.5571
MW	-53.5342	113.0683	119.7706	113.4375	115.7273
GE	-54.6201	113.2403	117.7085	113.4221	115.0130

Table 2 shows that the information criteria values are less compared to most of the models taken in consideration showing that proposed model fits better compared to most of considered models applied on real data set 1.

Fitted density function, empirical distribution function, the histogram, and estimated distribution function are all displayed for MUBD, EEP, MW, GEE, WE, and GE distributions in Figure 5.



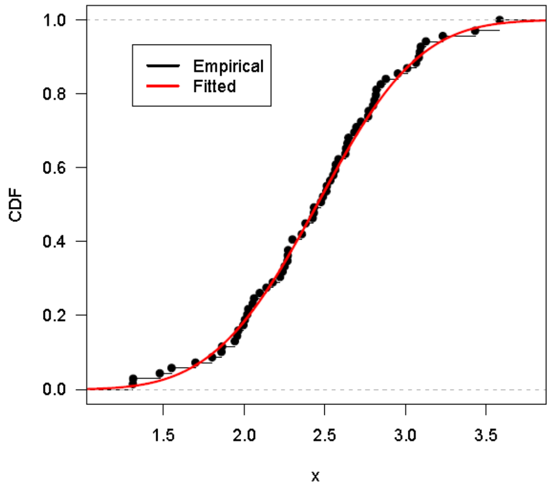


Figure 5. Empirical and estimated distribution functions (right panel) and the density function and histogram of fitted distributions (left panel)

Based on the results of the Kolmogorov-Smirnov (KS), Cramer-Von Mises (W) statistics and, Anderson-Darling (A^2), Table 3 compares the goodness-of-fit for MUBD model with other selected distributions. The MUBD distribution has a lower test statistic value and a higher p-value, which leads us to draw the conclusion that it provides findings that are more accurately fit to the distribution and more reliable than those from other distributions used as a comparison (table 3).

Table 3

The p-value related with statistics of goodness-of-fit

Distributions	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MUBD	0.0407(0.9998)	0.0165(0.9993)	0.1540(0.9983)
EEP	0.0366(0.9999)	0.0161(0.9994)	0.1455(0.9989)
MW	0.0834(0.7232)	0.1224(0.4867)	0.9061(0.4101)
GEE	0.0594(0.9678)	0.0502(0.8767)	0.3904(0.8578)
WE	0.0563(0.9809)	0.0347(0.9593)	0.2764(0.9546)
GE	0.0992(0.5060)	0.1733(0.3263)	1.1667(0.2804)

Table 3 shows that the test statistics are less and corresponding p-values are larger compared to most of the models taken in consideration showing that proposed model fits better compared to most of considered models applied on large real data set 1.

Dataset-2

The data set given below represents the number of major earthquakes (7.0+) from the United States Geological Survey (USGS) between 1990 and 2018 recorded are provided (USGS, 1990-2018).

18, 17, 13, 12, 13, 20, 15, 16, 12, 18, 15, 16, 13, 15, 16, 11, 11, 18, 12, 17, 24, 20, 16, 19, 12,19, 16, 7, 17

Log-Likelihood value, the LSE's, MLE's and CVME's for α, β, and λ are displayed in Table 4. Further we have calculated the Akaike information criterion (AIC) and Kolmogorov-Smirnov (KS) test statistics for three estimation approaches.

Table 4

Log-likelihood, Estimated parameters, AIC and Kolmogorov-Smirnov (KS) statistics

Estimation Method	alpha	beta	lambda	LL	AIC	KS(p-value)
MLE	1.4782	0.9971	55.0727	-77.2204	160.4407	0.1179(0.8147)
LSE	2.5576	0.7220	53.6955	-77.2808	160.5617	0.1020(0.9236)
CVME	3.6678	0.6370	56.2961	-77.5241	161.0482	0.1079 (0.8884)

Table 4 shows that parameters estimated by LSE and CVME fits the dataset 2 compared to MLE. The Q-Q and P-P plots of the MUBD model to the second data set is shown in Figure 6.

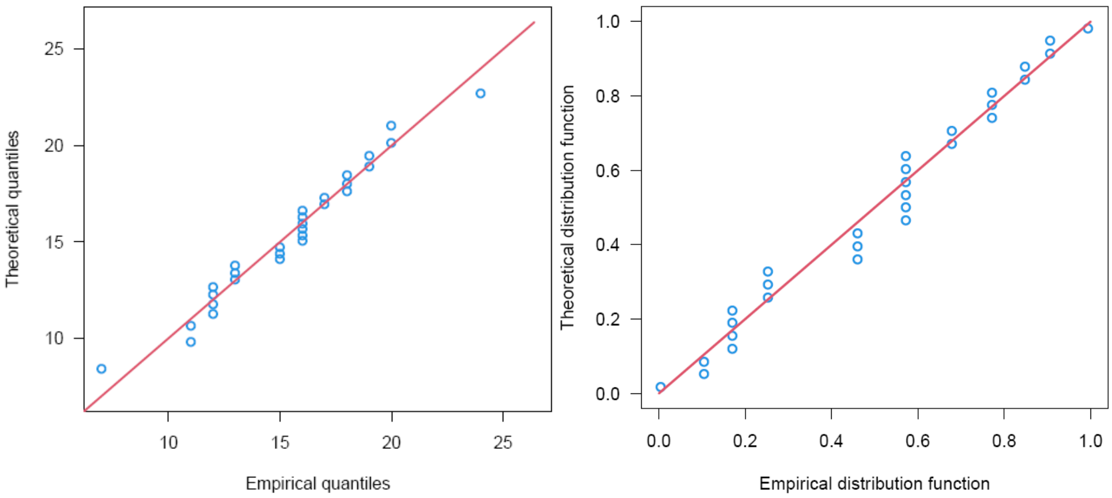


Figure 6. The Q-Q graph (right part) and P-P graph (left part) for MUBDmodel.

We have chosen a few well-known distributions for comparison in order to show the MUBD distribution's goodness of fit. These are Generalized Weibull Extension (GWE) (Sarhan and Apaloo, 2013), Logistic Inverse Exponential (LIE) Distribution (Chaudhary & Kumar, 2020), Generalized Exponential (GE) distribution (Gupta & Kundu, 1999) , Odd Lomax Exponential (OLE) distribution (Ogunsanya et al.,2019) and Exponentiated Generalized inverted Exponential (EGIE) distribution (Oguntunde et al., 2014).

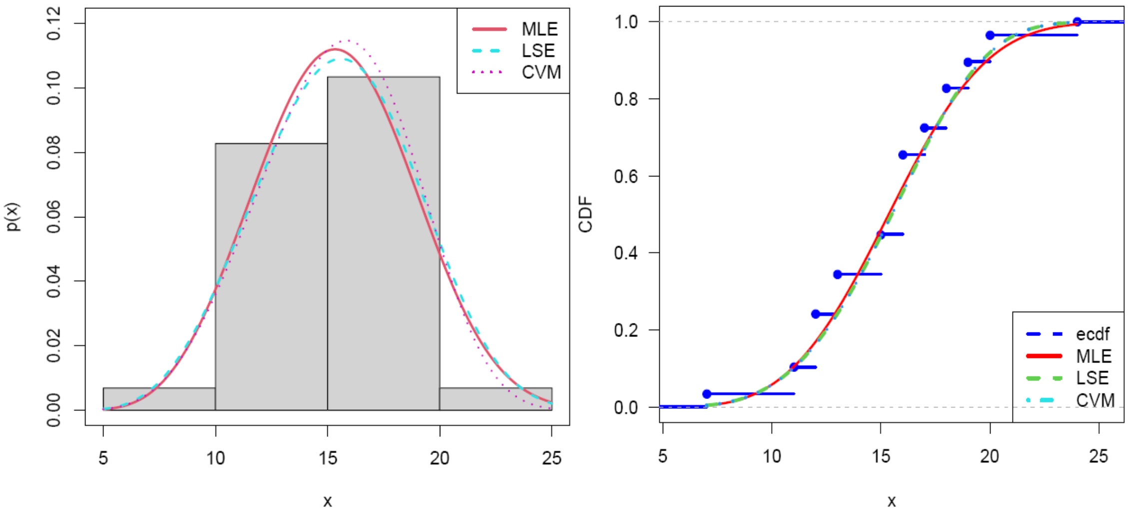


Figure 7.ECDF (right part) of MLE, LSE, and CVM, as well as a histogram and the density function of fitted distributions (left part).

In order to evaluate the potential of the proposed model, the values of the Corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Akaike information criterion (AIC), and Bayesian information criterion (BIC) are computed. The findings are displayed in Table 5.

Table 5

AIC, BIC, CAIC, HQIC, and Log-likelihood (LL) of MUBD distribution

Distributions	LL	AIC	BIC	CAIC	HQIC
MUBD	-77.2204	160.4407	164.5426	161.4007	161.7254
EGIE	-77.2329	160.4658	164.5677	161.4258	161.7505
GWE	-77.8310	161.6619	165.7638	162.6219	162.9466
LIE	-78.6688	161.3376	164.0721	161.7991	162.1940
GE	-79.8132	163.6263	166.3609	164.0878	164.4827
OLE	-81.8507	169.7014	173.8033	170.6614	170.9861

Table 5 shows that information criteria values are smaller than most of distributions taken in consideration indicating that proposed model fits small real dataset 2 better compared to existing distributions under consideration.

Fitted density function, empirical distribution function, the histogram, and estimated distribution function are displayed for the MUBD, EGIE, GWE, LIE, GE, and OLE distributions in Figure 8.

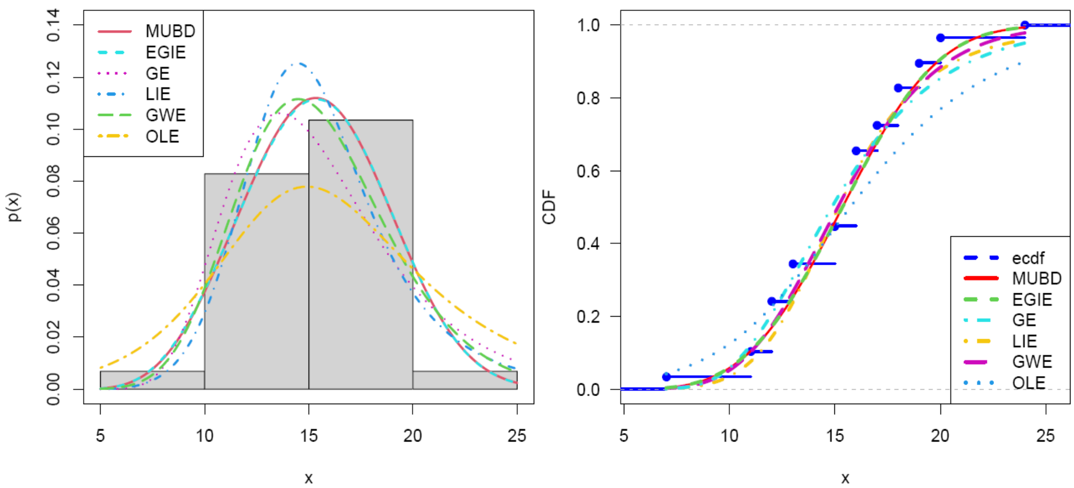


Figure 8. Empirical and estimated distribution functions (right panel) and the density function and histogram of fitted distributions (left panel)

Based on the results of the Kolmogorov-Smirnov (KS), Cramer-Von Mises (W) statistics and, Anderson-Darling (A^2), Table 3 compares the goodness-of-fit for MUBD model with other selected distributions. The MUBD distribution has a lower test statistic value and a higher p-value, which leads us to draw the conclusion that it provides findings that are more accurately fit to the distribution and more reliable than those from other distributions used as a comparison (table 6).

Table 6

The p-value related with statistics of goodness-of-fit

Distributions	***KS(p-value)***	***W(p-value)***	***A^2(p-value)***
MUBD	0.1180(0.8147)	0.0580(0.8303)	0.3759(0.8712)
EGIE	0.1181(0.8131)	0.0582(0.8294)	0.3774(0.8699)
GWE	0.1423(0.5995)	0.0817(0.6860)	0.4902(0.7555)
LIE	0.1429(0.5944)	0.0912(0.6331)	0.5847(0.6611)
GE	0.1668(0.3945)	0.1284(0.4651)	0.8118(0.4715)
OLE	0.1982(0.2048)	0.2353(0.2086)	1.5068(0.1750)

Table 6 shows that the test statistics are less and corresponding p-values are larger compared to most of the models taken in consideration showing that proposed model fits better compared to existing considered models applied on real data set 2.

5. Concluding Remarks

A new three-parameter modified upside down bathtub-shaped hazard function distribution is presented. The forms of the quantile function, skewness, and kurtosis, the probability density, cumulative density, hazard rate functions, and survival function, as well as other crucial statistical characteristics of the suggested distribution, are all explained. We have applied the approaches of CVME, LSE, and MLE to estimate the model parameters.According to our analysis, CVME and LSE outperform MLEs estimators by a significant margin.With the use of two real datasets (both large and small samples), the applicability, suitability, the superiority of

MUBD over other distributions are examined. It is found that this suggested distribution outperforms existing lifetime models. In the fields of survival analysis, probability theory, and practical statistics, we expect that this distribution will serve as an alternative.

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